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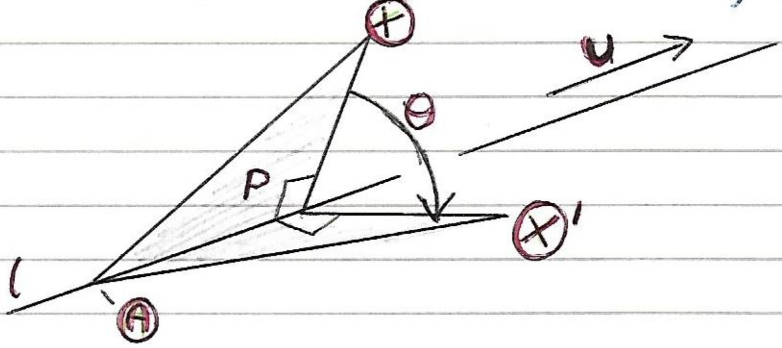
We shall begin by considering rotations in $\mathbb{R}^3$.
Suppose $X'$ is the image of $X$ under rotation about the line $l$ which passes through the point $A$ and is parallel to the unit vector $u$.
Suppose also that the angle of rotation is $\theta$ in a right hand screw sense about the direction of $u$ as shown below.




Let $P$ be the point on $l$ for which the plane $PXX'$ is perpendicular to $l$, and let $A$ be a given point on $l$.
We shall use the convention that $a$ is a posistion vector of $A$ etc.

We now get some practice in using scalar and vector products!

$$AP = ((x-a).u)u$$

Since $u$ is a unit vector ~

$$p = a + ((x-a).u)u \quad \leftarrow (7.2.1)$$

For convenience, we shall set

$$v = x - p \text{ and } v' = x' - p \quad \leftarrow (7.2.2)$$

Then

$$(7.2.3) \rightarrow v' = \cos\theta \, v + \sin\theta \, (u \times v)$$

Substituting from (7.2.2) into (7.2.3) we get

$$(7.2.4) \rightarrow x' = p + \cos\theta\,(x-p) + \sin\theta \{u \times (x-p)\}$$

Where $p$ is the vector defined in (7.2.1). This turns out to be

$$x' = a + ((x-a).u)u + \cos\theta\{u\times((x-a)\times u)\} + \sin\theta(u\times(x-a))$$

- with the notation above, since $P$ lies on $l$, then for some $\lambda \in \mathbb{R}$,

$$p = a + \lambda u$$

By considering the fact that, since $XP$ is perpendicular to $l$,

$$(x-p).u = 0$$

- Finding the value of $\lambda$ in terms of $x$, $a$ and $u$: we get

$$\lambda = (x-a).u$$

- Also with the notation above, suppose that $PX = PX' = r$.

- Finding $v.v'$ in terms of $r$ and $\theta$ we get:

$$v.v' = r^2\cos\theta$$

- Finding $v\times v'$ in terms of $r$, $\theta$ and $u$ we get:

$$v\times v' = r^2\sin\theta\, u$$

---

Find the image of a point $(1,2,3)$ under rotation through $\pi/3$ about the line $l$ passing through the point $(1,1,-1)$ and parallel to the position vector of $(1,0,-1)$.

(The angle is taken in a right hand screw sense about this vector)

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The solution to finding the image of the point $(1,2,3)$ under rotation through $\pi/3$ about line $l$ etc. is:

$$a = i + j - k \quad x = i + 2j + 3k \quad u = \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} k$$

now

$$x - a = j + 4k \text{ and } (x-a).u = -2\sqrt{2}$$

hence

$$((x-a).u)u = -2i + 2k \text{ and } p = -i + j + k$$

So that

(multiply)

$$x - p = 2i + j + 2k \text{ and } u \times (x-p) = \frac{1}{\sqrt{2}} i - 2\sqrt{2} j + \frac{1}{\sqrt{2}} k$$

So by substituting in (7.2.4), remembering that $\cos \pi/3 = 1/2$ and $\sin \pi/3 = \sqrt{3/2}$, we get:

$$x' = -i + j + k + \frac{1}{2}(2i + j + 2k) + \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{2}} i - 2\sqrt{2} j + \frac{1}{\sqrt{2}} k\right)$$

$$= \frac{\sqrt{6}}{4} i + \left(\frac{3}{2} - \sqrt{6}\right) j + \left(2 + \frac{\sqrt{6}}{4}\right) k.$$

In $\mathbb{R}^2$ we think of rotation about a point $P$, but again considering $\mathbb{R}^2$ as the plane $z = 0$ in $\mathbb{R}^3$, this would be equivalent to rotation about a line through the point $P$ parallel to $k$ (that is, perpendicular to the plane).

Instead of $u \times v$, we now think of vector $v^{\perp}$,

$$v^{\perp} = -bi + aj \text{ whenever } v = ai + bj$$

Considering $v^{\perp} = -bi + aj$ when $v = ai + bj$

The points A and P in the above working now coincide as the center of rotation and equation (7.2.4) is the most useful version of the required formula, with

$$(x-p)^{\perp} \text{ replacing } u \overset{\text{multiply}}{\times} (x-p)$$

- Finding the image of the point $(2,3)$ in the plane $\mathbb{R}^2$ under a rotation through $45°$ about the point $(1,1)$.

$$x = 2i + 3j \quad \text{and} \quad p = i + j$$

So that

$$x - p = i + 2j \quad \text{and} \quad (x-p)^{\perp} = -2i + j$$

therefore

$$x' = i + j + \frac{1}{\sqrt{2}}(i + 2j) + \frac{1}{\sqrt{2}}(-2i + j)$$

$$= \left(1 - \frac{1}{\sqrt{2}}\right) i + \left(1 + \frac{3}{\sqrt{2}}\right) j$$

(Remember that $1/\sqrt{2}$ is about $2/3$.)

A word on translations

**Translations** are very easy to deal with compared with the complications of reflections and rotations. We simply add a vector.

If the whole plane is translated we move all points by the same amount, in the same direction. So the same vector is added to the posistion vector of each point. Thus if $X'$ is the translation of $X$ with $A$ as origin then

$$x' = x + a$$

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Consider a reflection in a plane in $\mathbb{R}^3$. Suppose the point $X$ whose position vector is $x$ is reflected in the plane $\pi$, whose equation is:

$$(r-a).n=0$$

Where $a$ is the posistion vector of a point $A$ in the plane, and $n$ is a vector normal to the plane, and we shall choose $n$ to be a unit vector for convenience. Let $X'$ with position vector $x'$ be the image of $X$ under this reflection. Then line segment $XX'$ is perpendicular to $\pi$.

If this line segment cuts $\pi$ at $M$ we have

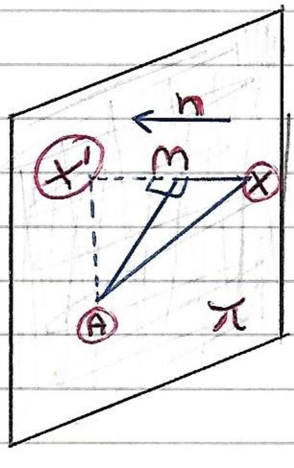
$$XM=((a-x).n)n$$

Since $n$ is a unit vector. Hence, as $XX'=2XM$

$$x'-x=2((a-x).n)n \quad \text{and}$$

$$x'=x-2((x-a).n)n \leftarrow (7.1.1)$$

- Now if the whole of $\mathbb{R}^3$ is reflected in the plane, we can find the image of any point in $\mathbb{R}^3$ by using equation (7.1.1)




$\boxed{n}$ = unit vector which is normal to the plane $\pi$

$\boxed{\pi}$ = plane

$\boxed{M}$ = where line cuts $\pi$

$\boxed{A}$ a point in plane $\pi$

$\boxed{X'}$ reflection of $X$

$\boxed{X}$ point outside of plane $\pi$

Note if $X$ lies on the plane $\pi$, then $X-a$ will be parallel to the plane and therefore perpendicular to $n$. This means that $(x-a).n=0$, and that $x'=x$.

So as expected, any point which is on the plane $\pi$, is not reflected and is invariant under it.

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- Finding the image of the point $(2,1,3)$ under reflection in the plane whose equation is

$$2x-2y+z=1$$

First we need a point on the plane, and $(0,0,1)$ satisfies the equation for the plane above.

Next a vector normal to the plane is

$$2i-2j+k$$

we divide this by it's length to obtain a unit vector in this direction

$$X=2i+j+3k, \quad a=k, \quad n=\frac{2}{3}i-\frac{2}{3}j+\frac{1}{3}k$$

Now by direct substitution into equation $(7.1.1)$ you can see that if $X'$ is the image of $X$ under this reflection

$$x'=\frac{2}{9}i+\frac{25}{9}j+\frac{19}{9}k.$$

So the image for the point $(2,1,3)$ is

$$(2/9,\ 25/9,\ 19/9)$$

We can check the previous page's result by showing that $x' - x$ is parallel to $n$.
That is, it is perpendicular to the plane, and also by finding the midpoint of $XX'$, showing that $XX'$ lies on the plane.

$$x' - x = \frac{8}{9}(2i - 2j + k)$$

which is clearly parallel to $n$.

Also

$$m = \frac{1}{2}(x' + x) = \frac{10}{9}i + \frac{17}{9}j + \frac{23}{9}k$$

These coordinates satisfy the equation for the plane, and so we have a double check that our image point is correctly found.

---

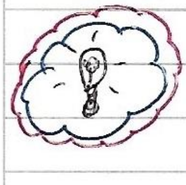
- Finding the image of point $(2, 0, -1)$ under reflection in the plane $x + y + z = 3$.

This would be $(10/3, \ 4/3, \ 1/3)$

where $x' - x = \frac{4}{3}n$

A midpoint of $XX'$ whose coordinates satisfy the equation of the plane would be:

$$(8/3, \ 2/3, \ -1/3)$$



The theory works equally well for $\mathbb{R}^2$. Although generally we consider reflection in a line $l$ in $\mathbb{R}^2$, we could think of the whole of $\mathbb{R}^2$ as a plane in $\mathbb{R}^3$ with $z = 0$.
We would then think of a reflection in a plane parallel to $k$ and containing $l$. Equation (7.1.1) would then still apply.

Previous pages here in this journal looked at three types of transformations of $\mathbb{R}^3$. This was because under such transformations distances and angles are invariant.

A reflection or a rotation or a translation does not change the size or shape of a body.

The invariance of size and shape depend on the invariance of the distances and angles.

## Definition 1

An Isometry of $\mathbb{R}^n$ is a function $t: \mathbb{R}^n \to \mathbb{R}^n$ for which all $x, y \in \mathbb{R}^n$,

$$|t(x) - t(y)| = |x - y|$$

## Theorem 1

If $t: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, then $t$ also preserves angles.

## Proof

Let $P, Q, R$ be any three distinct points in $\mathbb{R}^n$ such that $P', Q'$ and $R'$ are their respective images under $t$.

Then $P'Q' = PQ$, $Q'R' = QR$ and $R'P' = RP$,

so by the cosine rule:

$$\cos P'Q'R' = \frac{(P'Q')^2 + (R'Q')^2 - (P'R')^2}{2P'Q' \cdot R'Q'}$$

$$= \frac{PQ^2 + RQ^2 - PR^2}{2PQ \cdot RQ}$$

So angles are preserved under $t$.

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Note that on the previous page it is the size of the angle which is preserved.

In a reflection for example, angles are reversed, but since

$$\cos\theta = \cos(-\theta)$$

the previous page's equation is still satisfied as long as angles are reversed.

## Theorem 2

Translations, reflection and rotations are isometries of $\mathbb{R}^3$.

## Proof

① Consider the translation $t: \mathbb{R}^3 \to \mathbb{R}^3$ given by $t(x) = x + a$

Then $|t(x)-t(y)| = |(x+a)-(y+a)| = |x-y|$ and the condition is satisfied. The translation is an isometry.

② Let $t: \mathbb{R}^3 \to \mathbb{R}^3$ be a reflection. Then

$$|t(x)-t(y)| = |(x-y) - 2((x-y).n)n|$$

So
$$\begin{aligned}
|t(x)-t(y)|^2 &= \{t(x)-t(y)\}.\{t(x)-t(y)\} \\
&= (x-y).(x-y) - 4(x-y).\{((x-y).n)n\} + 4((x-y).n)^2 n.n \\
&= |x-y|^2 - 4|(x-y).n|^2 + 4|(x-y).n|^2 \\
&= |x-y|^2
\end{aligned}$$

So

$$|t(x)-t(y)| = |x-y|$$

...and a reflection is an isometry.

## Combinations of reflections, rotations and translations | Vectors

It is not too difficult to see that if we have a translation through $a$ followed by a translation through $b$, the resultant is a translation through $a+b$, since if:

$$x' = x + a \text{ and } x'' = x' + b$$

then $x'' = (x+a)+b = x + (a+b)$

- What happens when a reflection is followed by a reflection? This depends on the relationship between the two planes.

### ① Succesive Reflection in Two Parallel Planes

Consider the two planes $\pi$ and $\pi'$ whose equations are

$$(r-a)\cdot n = 0 \text{ and } (r-b)\cdot n = 0$$

Suppose reflection in $\pi$ takes $X$ to $X'$, and reflection in $\pi'$ takes $X'$ to $X''$. Then

$$x' = x - 2((x-a)\cdot n)n \text{ and } x'' = x' - 2((x'-b)\cdot n)n.$$

So $x'' = x - 2((x-a)\cdot n)n - 2[\{x - 2((x-a)\cdot n)n - b\}\cdot n]n$

$$= x - 2((a-b)\cdot n)n.$$

This is a translation by $2dn$ (two), where $d$ is the distance between the parallel planes.

This means a reflection in one plane followed by a reflection in a parallel plane, is a translation through a distance which is twice the distance between the two planes.
It is also a translation that is perpendicular to both planes.

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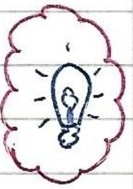
Combinations of reflections, rotations and translations | Vectors

## ② Successive Reflection in Two Non-Parallel Planes

It is also possible to show that a reflection in a plane $\pi'$, where $\pi$ and $\pi'$ are not parallel, is a rotation.

In this case it is actually a rotation about the line of intersection of the two planes. The angle of which is twice the angle that is between the two planes.



So we have found that both translations and rotations can be thought of as combinations of reflections.

It is also true to say that all isometries that are of $\mathbb{R}^3$, are generated by reflections within $\mathbb{R}^3$.

Another way of putting the above statement is to say that every isometry of $\mathbb{R}^3$ can be effected by a combination of reflections.

All of these isometries are much easier to deal with if the planes of reflection or the axes of rotation contain axes of coordinates.

This is where orthonormal bases come in useful.

Ex. □ For example. If we want the reflection of the vector $ai+bj+ck$ in the plane $z=0$. (Or the plane that contains the $x$ and $y$ axes.)

This is the vector

$$ai+bj-ck$$

In the same way if $\{u, v, w\}$ is an orthonormal basis of $\mathbb{R}^3$. The reflection of the point is the position vector $\alpha u+\beta v-\gamma w$.

Similar results are true for rotations

## Summary

① For a reflection.
If $a$ is the position vector of a point on the plane of reflection

$$x' = x - 2((x-a).n)n$$

② For a rotation.
If $a$ is the position vector of a point on the axis of rotation, and $u$ is a unit vector parallel to this axis

$$x' = a + ((x-a).u)u + \cos\theta$$
(continued) → $$\{u \times ((x-a) \times u)\} + \sin\theta(u \times (x-a))$$
(multiply)

③ A translation is given by

$$x' = x + a$$

④ An isometry of $\mathbb{R}^n$ is a function $t: \mathbb{R}^n \to \mathbb{R}^n$
Such that $t: \mathbb{R}^n \to \mathbb{R}^n$ =

$$\forall x, y \in \mathbb{R}^n, \ |t(x) - t(y)| = |x - y|$$

⑤ Reflections, rotations and translations - in any combination are isometries.

⑥ Any isometry of $\mathbb{R}^3$ can be regarded as a combination of reflections.

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Vector Space $\mathbb{R}^n$ | Vectors

- It is useful to consider with vectors the spaces beyond $\mathbb{R}^2$ and $\mathbb{R}^3 \rightarrow \mathbb{R}^n$.

Whereas in $\mathbb{R}^2$ a vector can be written in the form $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and in $\mathbb{R}^3$ the vector as $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

Within $\mathbb{R}^n$ we now have the form to consider $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

- note that $\mathbb{R}^n$ is sometimes seen as 'the set of points' with coordinates $(a_1, \ldots, a_n)$.

- Propeties of addition and scalar multiplication work for vectors in $\mathbb{R}^n$ in the ways below

$$\underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}}_{\text{(for addition)}} \text{ and } \underbrace{\alpha \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \alpha a_1 \\ \vdots \\ \alpha a_n \end{pmatrix}}_{\text{(mulliplication)}}$$

- Similary the length of the vector

$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ is $|a| = \sqrt{a_1^2 + \ldots + a_n^2}$, so a unit vector in the direction of $a$ is

$$\hat{a} = \frac{1}{\sqrt{a_1^2 + \ldots + a_n^2}} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

- The scalar product of two vectors in $\mathbb{R}^n$ is defined as

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \ldots a_n b_n$$

and we say that two non-zero vectors **a** and **b** are orthogonal in $\mathbb{R}^n$ if and only if their scalar product $\mathbf{a} \cdot \mathbf{b}$ is zero.

# Vector Space $\mathbb{R}^n$ | Vectors

There are some small differences in the algebra of vectors and ordinary algebra.
A collection together with their properties is given below.

For all vectors $u, v, w$ in $\mathbb{R}^n$,
all real numbers $\alpha$ and $\beta$

(a.1) ① $u + v \in \mathbb{R}^n$
So $\mathbb{R}^n$ is closed under vector addition

(a.2) ② $\mathbb{R}^n$ contains a zero vector $0$ such that
$v + 0 = 0 + v = v$

(a.3) ③ For each $v$ in $\mathbb{R}^n$ there is a vector $(-v)$ in $\mathbb{R}^n$ such that $(-v) + v = v + (-v) = 0$
(Every vector in $\mathbb{R}^n$ has an additive inverse in $\mathbb{R}^n$)

(a.4) ④ $v + (w + x) = (v + w) + x$ (vector addition is associative)

(a.5) ⑤ $v + w = w + v$ (vector addition is commutative)

(m.1) ⑥ $\alpha v \in \mathbb{R}^n$
This states that $\mathbb{R}^n$ is closed to scalar multiplication

(m.2) ⑦ $1v = v$

(m.3) ⑧ $\alpha(\beta v) = (\alpha\beta)v$

(m.4) ⑨ $(\alpha + \beta)v = \alpha v + \beta v$

(m.5) ⑩ $\alpha(v + w) = \alpha v + \alpha w$

Definition 1

NOTE

A set of vectors which satisfies the rules (a.1) to (a.5) and (m.1) to (m.5) is called a real vector space, and we refer to $\mathbb{R}^n$ as vector space $\mathbb{R}^n$.
There are vector spaces which are different to any $\mathbb{R}^n$ such as the vector space of all functions mapping $\mathbb{R}$ to $\mathbb{R}$ which has an infinite dimension.

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Subspaces of $\mathbb{R}^n$

Definition 2

A subspace of the vector space $\mathbb{R}^n$ is a non empty subset $S$ of $\mathbb{R}^n$ which is itself a vector space.

## Theorem 1

If $S$ is a non-empty subset of the real vector space $\mathbb{R}^n$, then it is a subspace of $\mathbb{R}^n$ only if

$$\forall u, v \in S, \quad u+v \in S \text{ and}$$

$$\forall u \in S \text{ and } \forall \alpha \in \mathbb{R}, \alpha u \in S$$

This means $S$ is closed under vector addition and scalar multiplication.
With those conditions satisfied, the rest of the conditions are also satisfied.

## Theorem 2

If $S$ is a non-empty subset of a real vector space $V$, then it is a subspace of $V$ if and only if

$$\forall u, v \in S, \text{ and } \forall \alpha \beta \in \mathbb{R}$$

Ex,

$$\alpha u + \beta v \in S$$

!

(Example) - Give a description for each of the following subsets of $\mathbb{R}^3$, and, in each case, determine if the subset is a subspace of $\mathbb{R}^3$.

① $S1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x+y-z=0 \right\}$

② $S2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x+y-z=1 \right\}$

(continued) → $S_3 =$ ③ $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 ; x^2 + y^2 + z^2 + 1 = 0 \right\}$

Solutions to $S_1, S_2$ and $S_3$ problems

Solution 1 →

① $S_1$ is a plane through the origin in $\mathbb{R}^3$. It's a non-empty set since

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in S_1$.

• Suppose $X_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, X_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \in S_1$ and $\alpha, \beta \in \mathbb{R}$

• Then $x_1 + y_1 - z_1 = 0$ and $x_2 + y_2 - z_2 = 0$

so it follows that $\alpha X_1 + \beta X_2 = \begin{pmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \\ \alpha z_1 + \beta z_2 \end{pmatrix}$

and $(\alpha x_1 + \beta x_2) + (\alpha y_1 + \beta y_2)$

$- (\alpha z_1 + \beta z_2)$

$= \alpha(x_1 + y_1 - z_1) + \beta(x_2 + y_2 - z_2)$

$= 0 + 0 = \underline{0}$ ← So $\alpha X_1 + \beta X_2 \in S_1$, making $S_1$ a subspace of $\mathbb{R}^3$

② $S_2$ is a plane passing through the point $(1,0,0)$, but not through the origin.

If $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S_2$, then $x + y - z = 1$.

Now $2x = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$, and $(2x) + (2y) - (2z) = 2 \neq 1$.

This meas that $2x \notin S_2$. So $S_2$ is closed for scalar multiplication, and not a subspace of $\mathbb{R}^3$.

③ $S_3$ is the empty set, as no real numbers can be $x, y, z$. A subspace of $\mathbb{R}^3$ must be a non-empty set of $\mathbb{R}^3$ - So $S_3$ is not a subspace of $\mathbb{R}^3$

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We now need to consider how to define subspaces of $\mathbb{R}^n$ in terms of vectors.

## Definition 3

Given the vectors $u_1, u_2 \dots u_n$, we say that any vector $v$ which can be written in the form

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$$

where $\alpha_1, \alpha_2 \dots \alpha_n \in \mathbb{R}$, is a linear combination of the vectors $u_1, u_2 \dots u_n$.

## Definition 4

If every vector in a vector space $V$ can be written as a linear combination of the vectors $v_1, \dots, v_n$ of $V$, then we say that the vectors $v_1, \dots, v_n$ span $V$.
Or that the set of vectors $\{v_1, \dots, v_n\}$ is a spanning set for $V$.

Ex.

## Example

Show that if $u = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $v = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $t = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

1. that $\{u, v, w\}$ is a spanning set for $\mathbb{R}^3$
2. but $\{u, v, t\}$ is not

## Solution

SOLUTION 1

1. Suppose that $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a general element of $\mathbb{R}^3$, and that $\rightarrow$

$\rightarrow x = \alpha \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\rightarrow$ Then

$$x = \alpha + 2\beta \quad (e1)$$
$$y = 2\alpha + \beta + \gamma \quad (e2)$$
$$z = 2\alpha + \beta. \quad (e3)$$

(continued) Multiplying (e3) by 2 and substracting it from (e1) gives

$$x - 2z = \alpha - 4\alpha \Rightarrow \alpha = \frac{1}{3}(2z - x)$$

and by back substitution leads to

$$\beta = \frac{1}{3}(2x - z) \text{ and } \gamma = y - z$$

and so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}(2z-x)\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \frac{1}{3}(2x-z)\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + (y-z)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

(You can check the results by confirming the components one at a time.)

- This shows that any element of $\mathbb{R}^3$ can be written as a linear combination of $u$, $v$ and $w$ and so $\{u, v, w\}$ is a spanning set for $\mathbb{R}^3$.

SOLUTION 2

(2) Suppose that $x = \alpha\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \beta\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \gamma\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Then $x = \alpha + 2\beta + \gamma$

$y = 2\alpha + \beta$ From this note that $y = z$,

$z = 2\alpha + \beta$ and because of this there would be no solution in $\alpha, \beta, \gamma$ for any vector for which the $y$ and $z$ components are different.

- For example the vector $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ in $\mathbb{R}^3$ cannot be expressed as a linear combination of $u, v, t$ so $\{u, v, t\}$ is not a spanning set for $\mathbb{R}^3$.

## Definition 5

The vectors $v_1, v_2, \dots, v_n$ are said to be linearly dependent if we can find scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

Note the zero vector is on the right.

If the vectors $v_1, v_2, \dots, v_n$ are not linearly dependent, they are therefore linearly independent.

## Definition 6

The vectors $v_1, v_2, \dots, v_n$ are said to be lineary independent if for scalars $\alpha_1, \alpha_2, \dots, \alpha_n$

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

which implies that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_n = 0$

## Definition 7

If $A = \{v_1, v_2, \dots, v_n\}$ is a set of vectors which are linearly independent. Then $A$ itself is a linearly indepent set. Otherwise it is linearly dependent.

Ex.

## Example

Show whether the set of vectors $\{u, v, w\}$ is linearly dependent, or linearly independent-

① $u = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

② $u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad w = \begin{pmatrix} 0 \\ -3 \\ -4 \end{pmatrix}$

(continued) Solution

Solution 1

① Suppose $\alpha u + \beta v + \gamma w = 0$ then

$$2\alpha + \beta + \gamma = 0 \quad (e1)$$
$$\alpha + \gamma = 0 \quad (e2)$$
$$-3\alpha + 2\beta + \gamma = 0 \quad (e3)$$

From the equation (e2) above, substituting this into (e1) and (e3) we get

$$\alpha + \beta = 0 \text{ and } -4\alpha + 2\beta = 0$$

The only solution to these equations is

$$\alpha = \beta = \gamma = 0$$

which means the three vectors are linearly independent

Solution 2

② Suppose $\alpha u + \beta v + \gamma w = 0$ then

$$\alpha + 2\beta = 0 \quad (e1)$$
$$2\alpha + \beta - 3\gamma = 0 \quad (e2)$$
$$3\alpha + 2\beta - 4\gamma = 0 \quad (e3)$$

From the first equation (e1) above, substituting this into (e2) and (e3), we get

$$-3\beta - 3\gamma = 0 \text{ and } -4\beta - 4\gamma = 0$$

So if $\gamma = 1$, $\beta = -1$ and $\alpha = 2$, all three equations are satisfied.

Since $\alpha, \beta, \gamma$ are not all zero, the three vectors are linearly dependent.

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## Geometrical Significance

If we have a single vector $x$ in $\mathbb{R}^3$, unless it is the zero vector, we can only have $\alpha x = 0$ when $\alpha = 0$.

This means a single non-zero vector is linearly independent.

The vectors would be linearly dependent if the set

$$\{v_1, \ldots, v_n\} \text{ where } v_i = 0$$

for some $i$, $1 \leq i \leq n$ then

$$\alpha_1 v_1 + \ldots, \alpha_n v_n = 0$$

would be satisfied when $\alpha_i = 1$ and $\alpha_j = 0$, $j \neq 1$.

So not all $a_k$ $(k = 1, \ldots, n)$ are zero, and the vectors would be linearly dependent.

This means that any set of vectors containing the zero vector is a linearly dependent set.

Suppose the vectors $v_1, v_2, \ldots v_n$ are linearly dependent. From the above definition, there exist real numbers $\alpha_1, \alpha_2, \ldots \alpha_n$ that are not all zero, so that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$$

We can think of $\alpha_1 \neq 0$ (by simply relabeling the vectors so this is true), giving

$$v_1 = -\frac{\alpha_2}{\alpha_1} v_2 - \ldots - \frac{\alpha_n}{\alpha_1} v_n$$

(continued) Now if $n=2$ this means that $v_1$ is a multiple of $v_2$, which means that $v_1$ is parallel to $v_2$.

In the same way, if $n=3$ then $v_1$ is a linear combination of $v_2$ and $v_3$, so $v_1$ is coplanar to $v_2$ and $v_3$.

So any two non-zero vectors in $\mathbb{R}^3$ are linearly independent if they are not parallel.
Also any three non-zero vectors are linearly independent, if they are not coplanar.

This is true in $\mathbb{R}^n$ for any $n$.
But if $n=2$ all vectors in $\mathbb{R}^2$ are coplanar, so there can't be anymore than two linearly independant vectors in $\mathbb{R}^2$.

In a similar way, we can have at most three linearly independent versions of $\mathbb{R}^3$, so at most $n$ within some $\mathbb{R}^n$.

So any spanning set for $\mathbb{R}^3$ must contain at least three vectors.
For if $u$ and $v$ are any two non-zero, non-parallel vectors in $\mathbb{R}^3$, then $u \times v$ is also a non-zero vector of $\mathbb{R}^3$.

---

Consider $u \times v = \alpha u + \beta v$

by taking the scalar product of $u \times v$ (which is perpendicular to both $u$ and $v$) with each side of the equation, we get -

$$(u \times v) \cdot (u \times v) = (u \times v) \cdot \alpha u + (u \times v) \cdot \beta v = 0$$

But the above line is a contridiction, as $(u \times v) \cdot (u \times v) \neq 0$ since $u \times v$ is a non-zero vector

(continued) So we have found a vector $u \times v$ in $\mathbb{R}^3$, which cannot be a linear combination of $u$ and $v$.
At least three vectors are needed to span $\mathbb{R}^3$.

Likewise, a spanning set of $\mathbb{R}^2$ contains at least two vectors, and more generally, a spanning set of $\mathbb{R}^n$ contains at least $n$ vectors.

NOTE! A spanning set can be found for $\mathbb{R}^3$ that contains more than three vectors.
For example, $\{i, j, k, i+j\}$. This is a spanning set for $\mathbb{R}^3$ containing four vectors.
However it is not linearly independent as a set.

$\{i, j\}$ is a linearly independent set of vectors for $\mathbb{R}^3$. But is not a spanning set in $\mathbb{R}^3$.

---

## Bases For Vector Spaces

We have seen already that the vectors $i$, $j$, and $k$ are important to the structure of $\mathbb{R}^3$.
These vectors are linearly independent which span $\mathbb{R}^3$.
Any vector in $\mathbb{R}^3$ can be written as a linear combination of these vectors. Each of these linear combinations is also unique.

These are not the only 3 vectors for which this is true as shown in the next definition.

### Definition 8

In a vector space $V$ the subset $B=\{v_1, v_2, \dots v_n\}$ is basis for $V$ if -

1. the vectors of $B$ are linearly independent, and
2. the vectors of $B$ span $V$

## Definition 9

We call $\{i, j, k\}$ the standard basis for $\mathbb{R}^3$.

## Theorem 3

A basis for $\mathbb{R}^n$ contains exactly $n$ vectors

A linearly independent set in $\mathbb{R}^n$ contains at most $n$ vectors, and a spanning set for $\mathbb{R}^n$ contains at least $n$ vectors.
So if $B$ is a basis for $\mathbb{R}^n$ it must contain at least $n$ vectors, and at most $n$ vectors.

## Theorem 4

① If $B = \{v_1, v_2, \ldots, v_n\}$ is a subset of $\mathbb{R}^n$, comprising $n$ linearly independent vectors.
Then $B$ is a basis for $\mathbb{R}^n$.

② If $S$ is a subspace of $\mathbb{R}^n$, and $B' = \{v_1, v_2, \ldots, v_k\}$ is a basis of $S$, containing $k$ vectors.
Then any set of $k$ linearly independent vectors of $S$, is a basis for $S$.

## Definition 10

We say that a subspace $S$ of $\mathbb{R}^n$ has dimension $k$ when there are $k$ vectors in a basis for $S$.
So $\mathbb{R}^n$ has dimension $n$.

In the above statement, the definition of dimension agrees with our useal idea of dimensions from our world. Where a line has dimension 1, a plane dimension 2 and three-dimensional space as dimension 3. A point would be zero dimensional.

Ex.

## Example

Show whether or not the following subsets of $\mathbb{R}^3$ are bases of $\mathbb{R}^3$

① $S_1 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \right\}$

② $S_2 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

SOLUTION 1

## Solution ①

Suppose $\alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ then

$$\alpha + \beta + 2\gamma = 0 \quad \text{eq 1}$$
$$2\alpha + 2\beta - \gamma = 0 \quad \text{eq 2}$$
$$\alpha + 3\beta + 2\gamma = 0 \quad \text{eq 3}$$

Adding $2\times$ eq2 first to eq1 and then to eq3 we get

$$5\alpha + 5\beta = 0 \text{ and } 5\alpha + 7\beta = 0$$

The only solution to these, and all three equations is $\alpha = \beta = \gamma = 0$.
This means all three vectors are lineary independent.

From theorem 4, since having 3 lineary independent vectors of $\mathbb{R}^3$, We have a basis for $\mathbb{R}^3$.

SOLUTION 2

## Solution ②

since...

$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ The three vectors are lineary dependent, and also cannot form a basis for $\mathbb{R}^3$.

## Definition 11

A basis $B$ of $\mathbb{R}^n$ is an orthogonal basis if the vectors of $B$ are mutually orthogonal.
- Any two distinct vectors of $B$ are orthogonal.

If $B = \{V_1, V_2, V_3\}$ is an orthogonal basis of $\mathbb{R}^3$. Then

$$V_1 \cdot V_2 = V_2 \cdot V_3 = V_3 \cdot V_1 = 0. \quad \text{eq}_1$$

In the above case, it is easier to write the general vector of $\mathbb{R}^3$ as a linear combination of those basis vectors.

---

Taking the scalar product of this equation

$$X = \alpha V_1 + \beta V_2 + \gamma V_3$$

from each side with $V_1$, and using $\text{eq}_1$ above, we get

$$X \cdot V_1 = \alpha V_1 \cdot V_1 + \beta V_2 \cdot V_1 + \gamma V_3 \cdot V_1 = \alpha V_1 \cdot V_1$$

Now $V_1 \cdot V_1 \neq 0$ since no basis vector can be the zero vector, and $V_1 \cdot V_2 = V_1 \cdot V_3 = 0$ from $\text{eq}_1$, so

$$\alpha = \frac{X \cdot V_1}{V_1 \cdot V_1}$$

---

In this way, the $\alpha$ is calculated without having to solve simultaneous equations.

By similar means taking the scalar product each side of the equation with $V_2$ and $V_3$, we find

$$\beta = \frac{X \cdot V_2}{V_2 \cdot V_2}, \quad \gamma = \frac{X \cdot V_3}{V_3 \cdot V_3}$$

Ex.

## Example

Below uses the Gram-Schmidt process to obtain three orthogonal vectors, with the ones given -

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad v_2 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad v_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

## Solution

Set $u_1 = v_1$ then $\lambda u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \frac{2}{6} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}.$$

By choosing $\lambda = 1/3$, we have $u_2 = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$.

The $\lambda$ factor

It's alot simplier to not have any fractions in the vector. This makes the following stage easier too. It's the reason we use the $\lambda$ in the above.

$$\mu u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2$$

$$= \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \frac{1}{21} \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$$

$$= \frac{27}{42} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}.$$

- By choosing $\mu = 27/42$ our three orthogonal vectors are -

$$u_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

## Summary

1. $\mathbb{R}^n$ satisfies the conditions for a vector space.

2. The subspaces of $\mathbb{R}^3$ are (a) $\mathbb{R}^3$ itself, (b) any plane through the origin, (c) any line through the origin, (d) the origin itself

   For $\mathbb{R}^2$ the subspaces are (a) $\mathbb{R}^2$ itself, (b) any line through the origin, (c) the origin itself

3. Linear Combination, Linear Dependence and Linear Independence were defined in definitions 5, 6, and 7 in the previous pages.

4. A basis for a vector space $V$ is a set $S$ of vectors of $V$ such that $S$ is both a linearly independent set and a spanning set for $V$.

5. The dimension of a vector space $V$ is the number of vectors in a basis of $V$, which is constant for $V$.

6. In an orthogonal basis, the vectors are mutually orthogonal, and the Gram-Schmidt process is one method for finding an orthogonal basis from a given basis.

7. A set of mutually orthogonal vectors is a linearly independent set of vectors.

8. An orthonormal basis for a vector space $V$, is an orthogonal basis whose vectors are all unit vectors.

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- **Definition 1**

A linear transformation $t: \underline{V} \to W$ is a function from a vector space $\underline{V}$ to a vector space $W$ which satisfies the following conditions for all $u$ and $v$ in $\underline{V}$ and all $\alpha \in \mathbb{R}$:

① $t(u+v) = t(u) + t(v)$ and

② $t(\alpha u) = \alpha t(u)$.

A similar definition combining both of these above follows -

- **Definition 2**

A linear transformation $t: \underline{V} \to W$ is a function from a vector space $\underline{V}$ to a vector space $W$ which satisfies the following conditions for all $u$ and all $v$ in $\underline{V}$ and all $\alpha, \beta \in \mathbb{R}$:

$$t(\alpha u + \beta v) = \alpha t(u) + \beta t(v).$$

- **Definition 3**

If either of the above Definition 1 or Definition 2 we have $V = W = \mathbb{R}^n$ then this is called a linear transformation of $\mathbb{R}^n$.

- **Linear Transformations of $\mathbb{R}^2$**

$\mathbb{R}^2$ Space

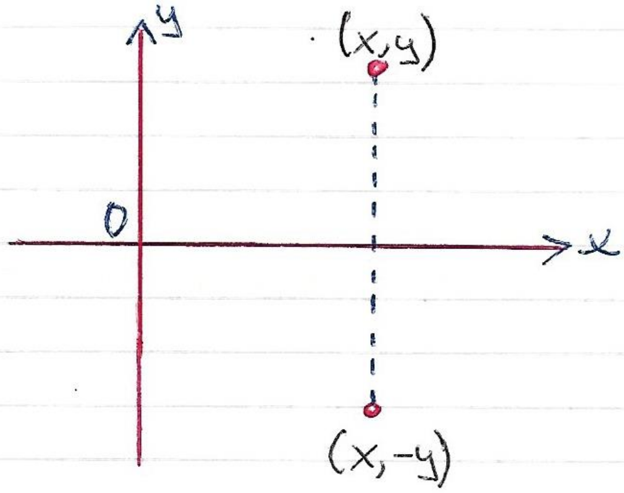
Let us start at looking at the linear transformations of $\mathbb{R}^2$, since these are simpler to explain and visualise geometrically.

One the next page we will begin with the first reflection of $\mathbb{R}^2$.

Linear Transformations 

reflection

Take a look at the reflection below of $\mathbb{R}^2$ that is in the $x$ axis.




We can see that the point whose position vector is $\begin{pmatrix}x\\y\end{pmatrix}$ is mapped onto the point with position vector $\begin{pmatrix}x\\-y\end{pmatrix}$ and we could write this as $\begin{pmatrix}x\\y\end{pmatrix}$ is mapped onto the point $\begin{pmatrix}x'\\y'\end{pmatrix}$ where

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}.$$

matrix transformations

Matrices will be used for the transformations that follow.

If $A = \begin{pmatrix}a & b\\c & d\end{pmatrix}$ is any $2\times2$ matrix with real entries, and if

$$t: \mathbb{R}^2 \to \mathbb{R}^2$$

is defined by

$$t(\cancel{x}) = Ax \text{ where } x = \begin{pmatrix}x\\y\end{pmatrix}$$

(x)

then by matrix algebra

$$A(x+w) = Ax + Aw \text{ and } A(\alpha x) = \alpha Ax.$$

This means that any matrix transformation of this type is a linear transformation.

With a linear transformation that maps $\begin{pmatrix}1\\0\end{pmatrix}$ to $\begin{pmatrix}a\\c\end{pmatrix}$, and $\begin{pmatrix}0\\1\end{pmatrix}$ to $\begin{pmatrix}b\\d\end{pmatrix}$.

we have $\begin{pmatrix}x\\y\end{pmatrix} = x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}$

and by using linear transformations, get

$$t\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = xt\left(\begin{pmatrix}1\\0\end{pmatrix}\right) + yt\left(\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$= x\begin{pmatrix}a\\c\end{pmatrix} + y\begin{pmatrix}b\\d\end{pmatrix}$$

$$= \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}.$$

This means that any linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a matrix transformation of the type described above.

By similar arguments, this leads to the next theorem

## Theorem 1

A function $t: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if it is a matrix transformation of the type

NOTE!

$$t(v) = Av$$ where $A$ is a real $m \times n$ matrix

Note

Note! A linear transformation $t: \mathbb{R}^n \to \mathbb{R}^m$ maps the zero vector of $\mathbb{R}^n$ to the zero vector of $\mathbb{R}^m$.

Since $t(v) = Av$ then $t(0) = A0 = 0$. From the above a linear transformation of $\mathbb{R}^2$ is $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, while for $\mathbb{R}^3$ this would be $\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$.

## Special Transformations for $\mathbb{R}^2$

Rotation

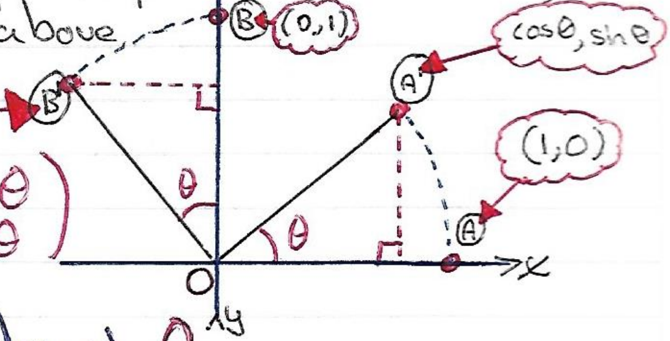
### Rotation about O

Under rotation through an angle $\theta$ about the origin (measuring $\theta$ anticlockwise from the positive $x$-axis), the images of the points $(1,0)$ and $(0,1)$ will be $(\cos\theta, \sin\theta)$ and $(-\sin\theta, \cos\theta)$ respectively.

The columns of the matrix representing the rotation are given by the position vectors of the points described above. Their matrix is

$-\sin\theta, \cos\theta$ B'

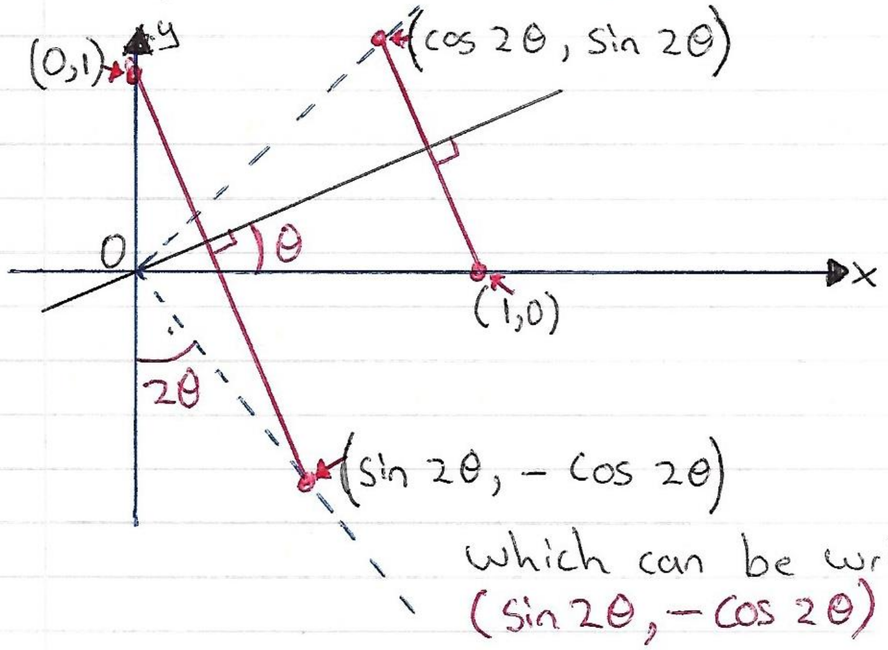
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$




Reflection

### Reflection in a Line through O

This is a reflection in a line making an angle $\theta$ with the $x$-axis




From the graph on the left point $(1,0)$ is mapped onto the point $(\cos 2\theta, \sin 2\theta)$, and point $(0,1)$ is mapped onto the point $(\sin(\pi-2\theta), \cos(\pi-2\theta))$ which can be written as $(\sin 2\theta, -\cos 2\theta)$

The matrix that represents the reflection in a line making an angle $\theta$ with the positive $x$ axis is -

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

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stretch

## Stretch Parallel to the Axes

Another type of familiar transformation of $\mathbb{R}^2$ is the stretch parallel to the x-axis.
In this case all distances from the y-axis are multiplied by a constant factor $\alpha$.
If $\alpha$ is positive the image stays on the original side of the y-axis.
while if $\alpha$ is negative, then the image shifts to the opposite side of the y-axis.

The $\alpha$ in these cases is called the scale factor of the stretch.

A stretch around the y-axis would have the matrix

$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ while a stretch that is also parallel to the y-axis, with

Scale factor $\beta$ would have the matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

with $|\alpha| < 1$ points will be closer to y-axis
while $|\alpha| > 1$ they will be further away from y-axis
The same applies for $\beta$ in respect of the x axis.

Enlarge/ Reduce

## Enlargement

A special case of this last transformation occurs when $\alpha$ and $\beta$ are the same. This will be an enlargement (or reduction if $<1$)

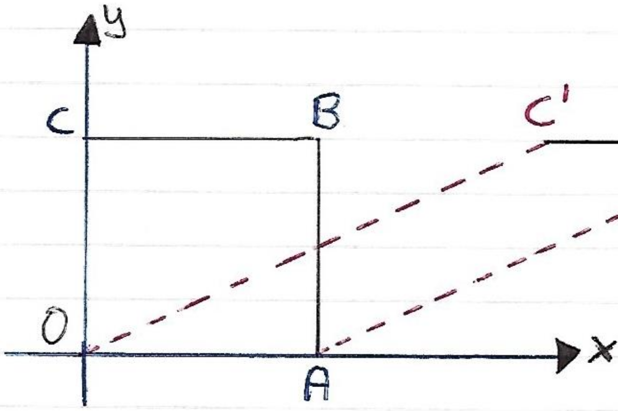
Since both scale factors are the same the matrix for an enlargement is

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

Shear

## Shear

A shear transformation keeps all points on one line through the origin fixed. But it moves all other points parallel to the fixed line.




The distance each point moves is proportional to it's distance from the fixed line. Opposite is where this line is fixed to the X-axis.

The matrix is in the form - $\begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix}$, $k \neq 0$ representing the transformation which sends any point $(x, y)$ to the point $(x + ky, y)$. The distance moved is proportional to the $y$ coordinate.

Identity

## Identity Transform

If in the case of shear we allow $K$ to be zero, then we get the Identity Matrix.

This represents the identity transformation, and leaves every point in $\mathbb{R}^2$ fixed.

It is also a special case of an enlargement (with $\alpha = 1$) and a rotation matrix where the angle turned through is any multiple of $2\pi$.

Single Transforms

## Singular Transformations

In all of the above transformations, a non-zero volume is mapped onto a non-zero volume.

This doesn't have to be so, such as the following translation $t$.

(continued) This transformation projects the whole of $\mathbb{R}^2$ onto a line in $\mathbb{R}^2$.

If $t$ has a matrix $\begin{pmatrix} pa & pb \\ qa & qb \end{pmatrix}$

then $\begin{pmatrix} pa & pb \\ qa & qb \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} pax + pby \\ qax + qby \end{pmatrix} = (ax+by)\begin{pmatrix} p \\ q \end{pmatrix}$.

In this case every point in $\mathbb{R}^2$ is mapped to a point on the line $py = qx$, and the plane collapses onto this line.

There is one single case amongst these, where

$p = q = 0$ or $a = b = 0$

in this the whole plane collapses to a single point at the origin.

A transformation of this type is called a singular transformation. This is because it's matrix is singular (that is, it has a zero determinant).

## Combinations of linear transformations

(combinations)

Suppose $t$ and $s$ are two linear transformations of $\mathbb{R}^2$ with respect to matrices $A$ and $B$.

Let $x'$ be the image of $x$ under $t$, and $x''$ be the image of $x'$ under $s$.

If we start with the $t$ transform, $x$ will be mapped to $x''$ because -

$$x' = t(x) = Ax \quad \text{and} \quad x'' = s(x') = Bx'$$

we get $x'' = B(Ax) = BAx$

This shows the combination transform $t$ followed by $s$, $(st)$, is a linear transform with matrix $BA$, $t$ starts first so is written on the right, and acts on the $x$ vector first. Then $s$ acts on $x'$. Written as $st(x) = s(t(x))$

## Facts About Previous Linear Transformations

1. **Succesive rotations** through angles $\theta$ and $\phi$ about origin result in a rotation about the origin by an angle of $\theta+\phi$

reflection + rotation → 2. **A reflection** in a line through origin at an angle $\phi$ followed by a rotation about origin through an angle $\theta$ results in a reflection in a line through origin at angle $\phi+\theta/2$

rotation + reflection → 3. **A rotation** about origin through an angle $\theta$ followed by a reflection in a line through origin at an angle $\phi$ results in a reflection in a line through origin at an angle of $\phi-\theta/2$

reflection + reflection → 4. **A reflection** in a line through origin at an angle $\phi$ followed by a reflection in a line through origin at an angle $\theta$ results in a rotation about the origin through angle of $2(\theta-\phi)$.

Ex.

## Example

The matrices for a reflection in a line through origin at an angle $\phi$ followed by a rotation about the origin through angle $\theta$.

$$\begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \text{ and } \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- The matrix for the reflection followed by a reflection is below

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos \phi \end{pmatrix}$$

(multiply)

$$= \begin{pmatrix} \cos\theta\times\cos 2\phi - \sin\theta\times\sin 2\phi & \cos\theta\times\sin 2\phi + \sin\theta\times\cos 2\phi \\ \sin\theta\times\cos 2\phi + \cos\theta\times\sin 2\phi & \sin\theta\times\sin 2\phi - \cos\theta\times\cos 2\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\phi+\theta) & \sin(2\phi+\theta) \\ \sin(2\phi+\theta) & -\cos(2\phi+\theta) \end{pmatrix}$$

This is the matrix for reflection in a line through origin at an angle $(\phi+\theta/2)$.

Two points determine a unique line containing them. So $[1,2,3]$ and $[3,2,1]$ lie on the line $x-2y+z=0$. In addition two lines meet at a unique point. So the lines $x+2y+3z=0$ and $3x+2y+z=0$ meet in the point $[1,-2,1]$.
If $A=(a,b,c)$ and $X=(x,y,z)$ then the line $ax+by+cz=0$ can be written as $A \cdot X = 0$ and the dual point is $[A]$.
This is represented in space as the line through the origin at right angles to the plane $ax+by+cz=0$.
Or to simplify calculations
by the notation $-\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1]$
Then the dual coordinates of the line through $[a_1,a_2,a_3]$ and $[b_1,b_2,b_3]$ are $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ which are also the coordinates of the intersection point of the lines $a_1x+a_2y+a_3z=0$ and $b_1x+b_2y+b_3z=0$.

- **Example**: to find the dual coordinates of the line through points $[1,1,-1]$ and $[2,0,1]$ consider
$\begin{vmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix} = [1-0, -2-1, 0-2] = [1,-3,2]$ The line equation is $x-3y-2z=0$.

## Quadric Surfaces - Equations

Ellipsoids

- Ellipsoids - These are two dimensional analogue of the ellipse. The standard example has the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All sections are ellipes. This is the only quadric surface which is bounded. So any plane sufficiently far from the origin will not meet the ellipsoid.
If $a=b>c$ the surface is called a geoid or oblate spheroid. Many planets under their own gravitational influence form geoids. A trigonometric parametric formula for a general point on the ellipsoid is $(a\cos\theta\sin\phi, b\sin\theta\sin\phi, c\cos\phi)$.
A rational parameterisation involving $s,t$ given by
$$\left(a\frac{2s}{1+s^2}\left(\frac{1-t^2}{1+t^2}\right), b\left(\frac{1-s^2}{1+s^2}\right)\left(\frac{1-t^2}{1+t^2}\right), c\frac{2t}{1+t^2}\right)$$

# Ordinary Itegrals of Vector Valued Functions



Vector Indefinite Integral

Let $R(u) = R_1(u)i + R_2(u)j + R_3(u)k$ be a vector depending on a single scalar variable $u$. Where $R_1(u)$, $R_2(u)$, $R_3(u)$ are assumed to be continuous in a specific interval, Then

$$\int R(u)\,du = i\int R_1(u)\,du + j\int R_2(u)\,du + k\int R_3(u)\,du$$

is called an indefinite integral of $R(u)$. If there exists a vector $S(u)$ such that $R(u) = \frac{d}{du}(S(u))$ then

$$\int R(u)\,du = \int \frac{d}{du}(S(u))\,du = S(u) + c$$

where $c$ is an arbitary constant vector independent of $u$. The definite integral between limits $u=a$ and $u=b$ can in such case be written

$$\int_a^b R(u)\,du = \int_a^b \frac{d}{du}(S(u))\,du = S(u) + c\Big|_a^b = S(b) - S(a)$$

This Integral can also be defined as a limit of a sum in a manner analogous to that of elementary integral calculus.

Example Suppose $R(u) = u^2 i + 2u^3 j - 5k$. Find (a) $\int R(u)\,du$ and (b) $\int_1^2 R(u)\,du$.

(a) $\int R(u)\,du = \int [u^2 i + 2u^3 j - 5k]\,du = i\int u^2\,du + j\int 2u^3\,du + k\int -5\,du$

$$= \left(\frac{u^3}{3} + c_1\right)i + \left(\frac{u^4}{2} + c_2\right)j + (-5u + c_3)k$$

$$= \frac{u^3}{3}i + \frac{u^4}{2}j + 5uk + c$$

where $c$ is a constant vector $c_1 i + c_2 j + c_3 k$.

(b) - from (a)

$$\int_1^2 R(u)\,du = \frac{u^3}{3}i + \frac{u^4}{2}j - 5uk + c\Big|_1^2 = [(8/3)i + 4j - 10k] - [-(1/3)i + (1/2)j - 5k]$$

$$= (7/3)i + (7/2)j - 5k$$

Vectors